Fundamental Concepts of Probability

Basic Definitions:-

* **Experiment**:- By an experiment, we mean any prosedure that:-

1- can be repeated, theoretically, an infinite number of times.

2- Has a well-defined set of possible outcomes.

* **Sample outcome**:- مخرجات التجربة، يعني لما أعمل التجربة لمرة واحدة، إيش هي النتيجة بتكون، {H,T} ، ويرمز له بـ (s) ← small (s)

* **Sample space**:- كُل مخرجات التجربة، ونرمز له بالرمز (S) ← capital S ، ولازم يحتوي كل شيء ينتج عن التجربة، ويكتب على شكل set (مجموعة).

* **Event**:- subset of the sample space.
It could be either the sample space, or the sample outcome, or even any subset of the sample space.

$A = \{H\}$
$B = \{T\}$
$C = \{H, T\} = S$
$D = \{\} = \phi$

Where A, B, C, D, are events for the experiment of flipping a coin.

Example :- Consider the experiment of flipping a coin three times.

[a] What is the sample space?

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

[b] Which sample outcomes make up the event:- A: Majority of coins show heads.

$A = \{HHH, HHT, HTH, THH\}$

* يمنع منعاً باتاً كتابة الـ event الواحد المرتبط زي هاد H,H,H لأنه هيك بالمرة الواحدة كانت البيضة فيها زي ما هي ممنوع *

example:- Consider the experiment of flipping two identical coins at the same time.

a) Write the sample space.

$S = \{HH, HT, TH, TT\}$

ممنوع منعاً باتاً :-

$S = \{TT, \cancel{HT}, \cancel{HH}\}$

وذلك لأننا خلال التعريف حكينا إنو الـ sample space هو كل مخرجات التجربة، ومعنى التانية إنو وقت ما بعرف رح يطلع معي TH ، إنما دائماً HT
a for coin 1, b for coin 2.

b) A: At least one head is observed.

$A = \{HT, TH, HH\}$.

c) B: Two tails are observed.

$B = \{TT\}$

D) C: Two heds and one tail are observed.

$C = \{\} = \phi$

Example:- Let us consider the experiment of tossing a dice.

a) determine the sample space.

$S = \{1, 2, 3, 4, 5, 6\}$

b) B: even number is observed.

$B = \{2, 4, 6\}$

c) C: odd number is observed.

$C = \{1, 3, 5\}$.

d) D: number is less than 5.

$D = \{1, 2, 3, 4\}$

e) E:- number is divisible by 3.

$E = \{3, 6\}$.

Algebra of events :-

Let A and B be two events defined over the sample space S, then:-

and = تقاطع

تقاطع

- The **intersection** of A and B, ($A \cap B$) is the events whose outcome belongs to both A and B.

. 2 events تقاطع بين

or = اتحاد

- The **union** of A and B, ($A \cup B$) is the event whose outcome belongs to either A **or** B **or** Both.

-: (tossing a dice) نأخذ أمثلة بحلها للمثال السابق

(f) F: $\underbrace{\text{even numbers}}_{B}$ $\underbrace{\text{Less than 5.}}_{D}$

$F = B \cap D$

$= \{2,4,6\} \cap \{1,2,3,4\}$

$= \{2,4\}$.

(g) $G = B \cap C$

$\{2,4,6\} \cap \{1,3,5\}$

$= \{\} = \phi$

(h) h: $\underbrace{\text{even number}}_{B}$ $\overset{\text{union "U"}}{\text{OR}}$ $\underbrace{\text{a number divisible by 3.}}_{E}$

$\therefore H = B \cup E = \{2,3,4,6\}$ → الترتيب مش مهم

عند كتابة الـ set طبيعي بوبط أكتب $\{2,4,6,3\}$

$\boxed{\text{لكن ممنوع التكرار}}$

(I) I: $\underbrace{\text{even}}_{B}$ or $\underbrace{\text{odd}}_{C}$ number is observed.

$I = B \cup C$

$I = \{1, 2, 3, 4, 5, 6\} = S$

- Events A and B are said to be **Mutually Exclusive** (or **disjoint**) if they have no outcomes in common, that is $A \cap B = \phi$, where ϕ is the null set (a set which contains no outcomes).

(J) Are B and D disjoint?

$B \cap D \overset{?}{=} \phi$

$B \cap D = \{2, 4, 6\} \cap \{1, 2, 3, 4\}$

$= \{2, 4\} \neq \phi$

So, they are Not disjoint.

(K) Are events B and C disjoint?

$B \cap C \overset{?}{=} \phi$

$\{2, 4, 6\} \cap \{1, 3, 5\} = \phi$

$\therefore$ B and C are disjoint.

- The complement of A (A^c or $\bar{A}$) is the event consisting of all outcomes in S other than those contained in A.

(L) $B^c = ?$ → ماهي العناصر الموجودة في الـ S ومش موجودة بـ B

$S = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6\}$, $\therefore B^c = \{1, 3, 5\}$

$\overline{}$ ← complement.

(m) $\overline{B \cap D} =$

$B \cap D = \{2, 4\}$

$\therefore \overline{B \cap D} = \{1, 3, 5, 6\}$

(n) $B \cap B^c = ?$

complement العكس

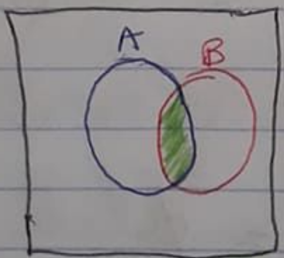
$B \cap B^c = \phi$ ← دااائماً وأبداً ، العنصر تقاطع (الضد) c يساوي فاي

(o) $B \cup B^c = ??$

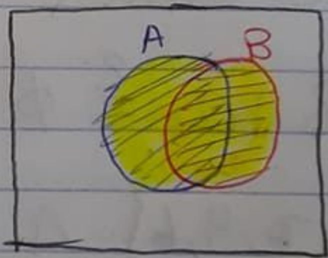
$B \cup B^c = S$.

- **Venn Diagram** is a graphical format often used to simplify the manipulation of complex events.

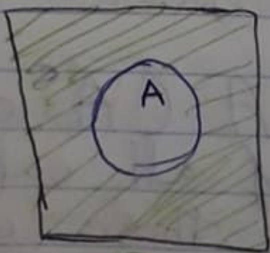
→ ex:-



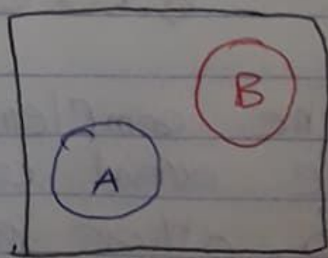
$A \cap B =$ ▇



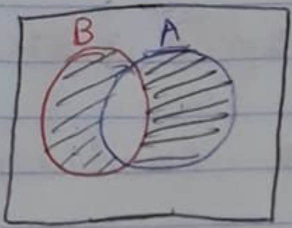
$A \cup B =$ ▇



$A^c =$ ▇



$A \cap B = \phi$



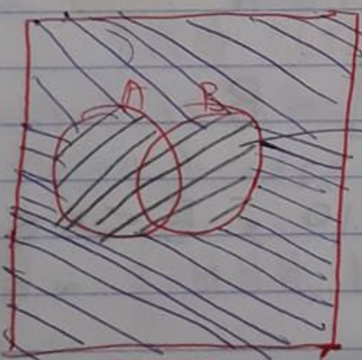
$(A\cap B^c)\cup(B\cap A^c) =$

examples:-

$\bar{A} =$, $\bar{B} =$

$\overline{A\cap B} =$ $\equiv$ $\bar{A}\cup\bar{B} =$

De Morgan's Law

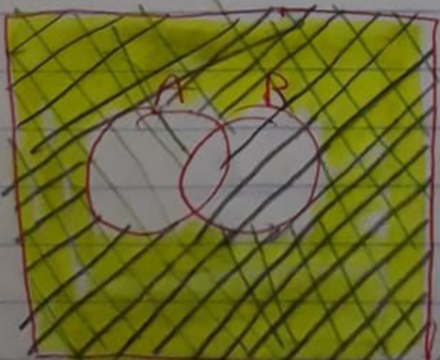


$A\cup B =$

$\overline{A\cup B} =$

$A^c =$

$\bar{B} =$



$\equiv$ De Morgan's Law.

$\bar{A}\cap\bar{B} =$

: De Morgan's Laws :-

1. $\overline{A \cap B} = \bar{A} \cup \bar{B}$

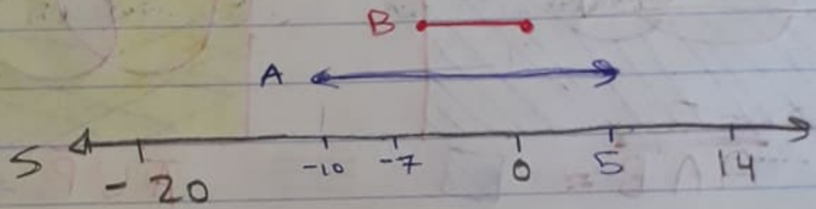
2. $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Example :- The sample space of an experiment is :-

$S = \{-20 \le x \le 14\}$. If $A = \{-10 \le x \le 5\}$ Real numbers

and $B = \{-7 \le x \le 0\}$. Find :- Notice :- $x \in R$

1. $A \cup B$



$A \cup B = \{-10 \le x \le 5\} = A$

2. $A \cap B = \{-7 \le x \le 0\} = B$

3. $\overline{A \cup B} = \{-20 \le x < -10 \ , \ 5 < x \le 14\}$

Probability :-

Definitions of Probability :-

1 Classical (apriori)

If the sample space S of an experiment consists of finitely many outcomes (points) that are equally likely, then the probability of event A, P(A) is:-

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } S}$$

$P(S)=1$, $P(\phi)=0$

→ ex:- consider the experiment of flipping a fair coin for 3 times.

$P(H) = P(T) = 50\%$

a) what is the probability of observing three heads?

$S = \{HHH, HHT, HTH, THH, THT, TTH, TTT\}$

$A = \{HHH\}$

for ITC

$\therefore P(A) = \frac{1}{8} = \underline{0.125} = \underline{12.5\%}$

ممنوع نكتب (%) بالامتحان لأنه السي عشان يصحح صح، عشان ما بيقرأ على ال %.

b) what is the probability of observing at lest two tails?

$B = \{TTH, THT, HTT, TTT\}$

$P(B) = \frac{4}{8} = \frac{1}{2} = \boxed{0.5} = 50\%$

right for ITC

بعد ما أقوم بالتجربة بقرر شو ال probability

2 Relative Frequency (a posteriori)

Let an experiment be repeated (n) times under identical conditions, then the relative frequency :- كثير كثير من المرات

$$P(A) = \lim_{n \to \infty} \frac{f(A)}{n} = \frac{\text{Number of times A occurs}}{\text{Number of trials}}$$

$$0 \leq \frac{f(A)}{N} \leq 1 ,$$

→ ex:- consider the experiment of flipping a coin, Assume the coin is flipped for 10^{10} times and head is observed for 0.5×10^{10} times. what is the probability of head? Using the Relative frequency.

$$P(H) = \frac{0.5 \times \cancel{10^{10}}}{\cancel{10^{10}}} = 0.5$$

ex:- Consider the experiment of flipping a coin.

a) What is the probability of observing a head?

لما إنو مش معطيني إنو التجربة تكررت وولا بدي أي معلومة، فخياري الوحيد هو استخدام الـ classical.

$$S=\{H,T\} \quad , \quad A=\{H\} \, , \, P(A)=\frac{1}{2}=0.5$$

b) Assume the coin is flipped for 10^{12} times and head is observed in all trials, what is the probability of observing a head in a new trial?

∴ Using the Relative frequency:-

$$P(A)=\frac{f(A)}{n}=\frac{10^{12}}{10^{12}}=1=100\%$$

هاد يدل على أنو الـ coin تبعتنا وجهيها هما عبارة عن Head.
لأنو مستحيل 10^{12} مرة ولا مرة تصدف فيها يجي Tail.

$$\therefore P(B)=0\%$$

لـ Tail

لما أبعث مرور الـ bits بالهواء (wifi و wireless مثلا) بيتعرضوا لـ noise فمثلاً الإشارة بدل ما يكون (0) بيتحول لـ (1) أو 1 إلى (0)

...10101 | 0100 | 1100 | 0110 → 10110 | 00

ex:- In digital data transmission, the bit error Probability is (P). If 10000 bits are transmitted over a noisy communication channel and 5 bits were found to be in error, Find the bit error Probability (P)

$$P=\frac{5}{10000}=0.0005$$

$\boxed{3}$ Subjective :- probability is defined as a person's measure of belief that some given event will occur.

ليس لها مقياس معين، وتعتمد على وجهة نظر كل شخص بناءً على الظروف المحيطة.

- مثال :- ماهي احتمالية قطع الكهرباء خلال اليوم؟ كل هدا رح يعول نسبة معينة مختلفة عن الآخر بناءً على المعطيات الخاصة به والظروف المحيطة.

- مثال :- ماهي احتمالية نجاحك في هذه المادة؟ برضو رح تختلف من شخص لآخر.

$\boxed{4}$ Axiomatic هاد مش بيعرف الـ probability بشكل جديد إنما هو يضع مبادئ، وهاي المبادئ لازم نلتزم فيها لما نحكي عن الـ probability. وهاي الـ axioms مابلزم إلها proof.

Given a sample space (S), with each event (A) of (S) (subset of S) there is associated a number P(A), called the probability of (A), such that the following axioms of probability are satisfied :-

ومعناها إنو الحدث A مش موجود S.

$\boxed{1}$ $P(A) \geq 0$: Probability is nonnegative

$\boxed{2}$ $P(S) = 1$:- Probability of the sample space is certain.

$\boxed{3}$ For the mutually exclusive events (A) and (B) ($A \cap B = \phi$) :- $P(A \cup B) = P(A) + P(B)$;

ومعناه هاي النقطة مبدأ منطقي

$(A \cap B = \phi)$

→ ex:- $S = \{1,2,3,4\}$, $A = \{1,4\}$, $B = \{3\}$,
$C = \{1,3\}$.

a) $P(A \cup B) = ??$ ⇒ $A \cap B \overset{?}{=} \phi$ ⇒ $\phi = \phi$ ✓

$\therefore P(A \cup B) = P(A) + P(B)$

$= \frac{2\times 1}{4\times 2} + \frac{1}{4}$

$= \frac{3}{4}$

or $P(A \cup B) = P(\{1,4\} \cup \{3\})$

$= P(\{1,3,4\})$

$= \frac{3}{4}$.

(b) $P(A \cup C)$ ⇒ are A and C disjoint?

① $A \cap C = \{1\}$, so they are Not disjoint.

② so $P(A \cup C) = P(\{1,4\} \cup \{1,3\})$ نظرية نقطة 3 ما تربط

$= P(\{1,3,4\}) = \frac{3}{4}$

ex:- let A and b be two events defined on the sample space S. Assume $P(A \cup B) = 0.9$ and $P(A) = 0.4$. Determine $P(B)$ assuming A and B are disjoint. ⇒ هاد مفتاح الحل يعني

As $A \cap B = \phi$

$\therefore P(A \cup B) = P(A) + P(B) \Rightarrow 0.9 = 0.4 + P(B)$

$\therefore \boxed{P(B) = 0.5}$

$\boxed{4}$ If (S) is infinite (has infinitely many points), axiom (3) is to be replaced by:

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

Where $A_1, A_2, A_3 \ldots$ are mutually exclusive ($A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, $A_2 \cap A_3 = \phi$, ...).

ex → $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{6\}$, $D = \{4, 6\}$

$\boxed{a}$ $P(A \cup B \cup C) = ?$

Are A, B and C mutually exclusive (Disjoint)?

$A \cap B \overset{?}{=} \phi \Rightarrow \{1,2\} \cap \{3,4\} = \phi$ ∴ disjoint ✓

$A \cap C = \{1,2\} \cap \{6\} \overset{\checkmark}{=} \phi$

$B \cap C = \{3,4\} \cap \{6\} \overset{\checkmark}{=} \phi$

∴ yes, they are mutually exclusive.

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$= \frac{2}{6} + \frac{2}{6} + \frac{1}{6}$$
$$= \frac{5}{6}.$$

(b) $P(A \cup B \cup D) = ?$

$A \cap B = \phi$?
$A \cap D = \phi$
$B \cap D = \{4\}$

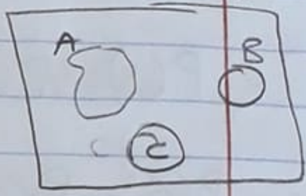
They Are Not disjoint

وبحل بالطريقة العادية زاي . باخد $A \cup B \cup D$ وبوجد الـ Probability

$$= P(\{1, 2, 3, 4, 6\}) = \frac{5}{6}$$

Caution: تحذير
$A \cap B \cap D = \phi \rightarrow$ doesn't indicate to anything, but we care about A, B, and D to be disjoint (Pairs → AB, BD, AD

ex → A, B, and C are three disjoint events defined over the sample space S. Assume $P(A) = 0.1$, and $P(B) = 0.4$, and $P(A \cup C) = 0.5$



$\boxed{a}$ what is $P(C) = ??$

A and C are disjoint

$\therefore P(A \cup C) = P(A) + P(C)$

$0.5 = 0.1 + P(C)$

$\therefore P(C) = 0.4$.

$\boxed{b}$ $P(A \cup B \cup C) = ??$ A, B, C are disjoint.

$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$= 0.1 + 0.4 + 0.4$

$= 0.9$

$\boxed{c}$ $P(A \cup B) = ??$ A and B are disjoint

$\therefore P(A \cup B) = P(A) + P(B)$

$= 0.1 + 0.4$

$= 0.5$

Basic Theorems for Probability:

$\boxed{1}$ $P(A^c) = 1 - P(A)$

Proof :- $S = A \cup A^c$

$P(S) = P(A \cup A^c)$, but $A \cap A^c = \phi$, disjoint

$\therefore P(S) = P(A) + P(A^c)$.

$P(S) = 1$
معناها بحل كثير

$1 = PA + P(A^c) \Rightarrow \boxed{P(A^c) = 1 - P(A)}$

[2] $P(\emptyset) = 0$, Proof:-

sample space → $S = S \cup S^c \rightarrow \emptyset$

$S = S \cup \emptyset$; $S^c = \emptyset$

$\boxed{S}$ $\therefore S^c = \emptyset$

$\therefore P(S) = P(S \cup \emptyset)$, S & S^c are disjoint

$\therefore \cancel{P(S)} = \cancel{P(S)} + P(\emptyset)$

$\therefore \boxed{P(\emptyset) = 0}$

[3] $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. ← هاد بشكل عام، سواء اكانوا A,B disjoint او لا .

[4] If A , B , and C are three events, then:-

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) .$$

→ البروبابلتي للتقاطع كل 2 مع بعض $\boxed{-}$ (عكس الإشارة) الـ Probability لكل واحد منهم

الـ Probability لتقاطع الـ 3 events $\boxed{+}$ يرجع بعكس الإشارة

→ $P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D) - P(A \cap B) - P(A \cap C) - P(A \cap D) - P(B \cap C) - P(B \cap D) - P(C \cap D) + P(A \cap B \cap D) + P(A \cap B \cap C) + P(A \cap C \cap D) + P(B \cap C \cap D) - P(A \cap B \cap C \cap D)$.

ex:- One integer is chosen at random from the numbers $\{1, 2, \ldots, 50\}$. What is the probability that the chosen number is divisible by 6? Assume all 50 outcomes are equally likely.

$S = \{1, 2, 3, \ldots, 50\} \rightarrow 50$ sample outcome in S.

$A = \{6, 12, 18, 24, 30, 36, 48\} \rightarrow 8$ sample outcomes in A

$$P(A) = \frac{\text{Number of elements in } A}{\text{Number of elements in } S} = \frac{8}{50} = 0.16$$

ex:- If the probability of occurrence of an even number is twice as likely as that of an odd number in the previous example. Find P(A); A is defined in the previous example.

$P(S) = P(\text{even}) + P(\text{odd}) = 1$

Let P be the probability of occurrence of an odd number, then (2P) will be the probability of occurrences of even number.

$$P(S) = 1 = P(\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \ldots \cup \{50\}).$$

$$1 = P(\{1\}) + P(\{2\}) + \ldots + P(\{50\}).$$

$$1 = P + 2P + P + 2P + \ldots + 2P$$

$$1 = 25P + 25(2P)$$

$$1 = 75P \Rightarrow \boxed{P = \frac{1}{75}}$$

عدد الـ positive element بالـ event A . الـ Probability للعدد الزوجي الوامر.

$$\therefore P(A) = 8 * 2P$$

$$= 8 \times \frac{2}{75}$$

$$= \frac{16}{75}$$

ex: suppose that a company has 100 employees who are classified according to their marital status and according to whether they are college graduates or not. It is known that 30% of the employees are married, and the percent of graduate employees is 80%. Moreover, 10 employees are neither married nor graduates, what proportion of married employees are graduates?

يعني بده التقاطع

Let M: set of married employees.
G: set of Graduate employees.
$N(\cdot)$: number of employees in any set $(\cdot)$.

$N(S) = 100$.
$N(M) = 0.3 \times 100 = 30$
$N(G) = 0.8 \times 100 = 80$

$N(M \cup G)^c = 10$ ← لا متزوج ولا متخرج

$N(M \cup G) = 100 - 10 = 90$

$N(M \cup G) = N(M) + N(G) - N(M \cap G)$ ← الي بدنا اياه

$90 = 30 + 80 - N(M \cap G)$

$\therefore N(M \cap G) = 20 \rightarrow \frac{2}{3}$ the married

→ employees are Graduates.

ex:- An experiment has two possible outcomes; the first occures with probability (P), the second with probability (P^2), find P.

$S = \{a, b\}$, $P(a) = P$, $P(b) = P^2$

$P(S) = P(\{a\} \cup \{b\})$

$= P\{a\} + P(\{b\})$

$P + P^2 = 1$

$P^2 + P - 1 = 0$

$\therefore P = \dfrac{-1 \pm \sqrt{5}}{2}$ (only the positive root is taken)

لأن الـ probability دائمًا موجبة أو صفر

so $\boxed{P = \dfrac{-1 + \sqrt{5}}{2}}$

Discrete Probability functions:-

If the sample space generated by an experiment contains either a finite or a countable infinite number of outcomes, then it is called a discrete sample space.

ex:- "S" = $\{1, 2, 3, 4, 5, 6\}$ is countably finite.

"S" = $\{1, 2, 3, \ldots\}$ is countably infinite.

~~Exam~~

Example:- Consider the experiment of tossing a dice.

a) write the sample space.

$S = \{1, 2, 3, 4, 5, 6\}$.

b) what is the probability of observing an even number.

$A = \{2, 4, 6\}$.

$P(A) = \frac{3}{6} = \frac{1}{2} = 0.5$.

c) what is the probability of observing a number divisible by 3.

$B = \{3, 6\} \Rightarrow P(B) = \frac{2}{6} = \frac{1}{3}$

D) what is the probability of observing a number less than 5?

$C = \{1, 2, 3, 4\} \Rightarrow P(C) = \frac{4}{6} = \frac{2}{3}$

e) Assume an even number is observed, what is the probability that it is divisible by 3.

S_{new}

I) $S_{new} = \{2, 4, 6\}$

$B_{new} = \{6\}$

$\therefore P(B_{new}) = \frac{1}{3}$

II) $P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1/6}{3/6} = \frac{1}{3}$

$P(\{6\})$

given (even number)

given

تُقرأ (B) given A

يعني إيش احتمالية B تحدث إذا حدث A

(8) Assume the number observed is less than 5, what is the Probability that it is divisible by 3.

$$P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{1/6}{4/6} = \frac{1}{4}$$

(9) Are A and B S. Independent?

* Two events are statistically independent (A) and (B), if :-

$$P(A \cap B) = P(A)P(B).$$

So $P(A \cap B) \overset{?}{=} P(A)\,P(B)$

$$\frac{1}{6} \overset{?}{=} \frac{1}{2} \times \frac{1}{3}$$

$\frac{1}{6} \overset{\checkmark}{=} \frac{1}{6}$, so A & B ar S.I.

So $\Rightarrow$ $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\cancel{P(A)}\,P(B)}{\cancel{P(A)}} = P(B)$

if

if A and B are S.I.

So the occurrence of A didn't affect the probability of B, and vice versa.

يعني يرضو

$$P(A/B) = P(A).$$

(11) Are B and C statistically Independent?

$P(B \cap C) \overset{?}{=} P(B)\,P(C)$ $\left\{ \frac{1}{6} \therefore \begin{array}{l} P(B/C) \neq P(B) \quad \frac{1}{3} \\ P(C/B) \neq P(C) \quad \frac{4}{6} \end{array} \right.$

$$\frac{1}{6} \overset{?}{=} \frac{2}{6} \times \frac{4}{6}$$

$\frac{1}{6} \neq \frac{8}{36}$ $\Rightarrow$ So they Are Not Statistically Independent.

Definition:- Statical independence:-

Two events (A) and (B) are said to be statistically independent, if:-

$P(A \cap B) = P(A)\,P(B)$, so we conclude that:-

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\,\cancel{P(B)}}{\cancel{P(B)}} = P(A)$$

$\Rightarrow$ a posteriori Probability = a priori Probability.

& $$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)\,\cancel{P(A)}}{\cancel{P(A)}} = P(B).$$

Hence, That means that the probability of (A) doesn't depend on the occurance or nonoccurance of (B) and vice versa. Hence, the given information does not change our initial perspection about the two given probabilities.

Independence of Three Events:-

Events (A), (B), and (C) are independent, if all the following conditions are satisfied:-

$$P(A \cap B) = P(A)\,P(B)$$

$$P(A \cap C) = P(A)\,P(C)$$

$$P(B \cap C) = P(B)\,P(C)$$

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C).$$

Theorem: Multiplication Rule (Chain Rule)

We know: $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$ ← For 2 events

ضرب تبادلي

$$\therefore \boxed{P(A \cap B) = P(A/B)\, P(B)}$$

& $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

$$\therefore \boxed{P(A \cap B) = P(B/A)\, P(A)}$$

$$\therefore \boxed{P(A \cap B) = P(A/B)\, P(B) = P(B/A)\, P(A).}$$

Multiplication Rule ; where $P(A) \neq 0$, $P(B) \neq 0$

→ for 3 events :-

$$P(A \cap B \cap C) = P(A)\, P(B/A)\, P(C/B,A)$$

معناها كالتالي :- P لواحد منهم فليكن A ← $P(A)$ ، ضرب
P لواحد آخر فليكن B given اللي عملناه قبله اللي هو A ← $P(B/A)$ ، ضرب
الجديد ↑ اللي قبله ↑

P للي فضل (اللي هو C) ، given الـ 2 اللي قبله B و A ، يعني
$P(C/B,A)$ وتُقرأ :- P, C given A and b ،
والفاصلة معناها آند .

Example:- A certain computer becomes inoperable if two components A and B both fail. The probability that A fails is 0.001, and the probability that B fails is 0.005. However, the probability that B fails increases by a factor of 4 if A has failed. Calculate the probability that :-

$\boxed{a}$ The computer $\underbrace{\text{becomes inoperable}}_{}$. $P(A \cap B)$

$P(A) = 0.001$

$P(B) = 0.005$

$P(B/A) = 4 \times P(B) = 4 \times 0.005 = 0.020$

$P(A \cap B) = \underbrace{P(A/B)}_{\text{مجهول}} P(B)$ OR $\underbrace{P(A \cap B)}_{\text{مطلوب}} = \underbrace{P(B/A)}_{\text{معلوم}} \underbrace{P(A)}_{\text{معلوم}}$

$\therefore P(A \cap B) = P(B/A)\, P(A)$

$= (0.02)(0.001)$

$= 2 \times 10^{-5}$

(b) "A" will fail if B has failed. → $P(A/B)$

أوجدناها في (a)

$P(A/B) = \dfrac{P(A \cap B)}{P(B)_{\text{معلوم}}} = \dfrac{2 \times 10^{-5}}{0.005} = \dfrac{2 \times 10^{-5}}{5 \times 10^{-3}} = 0.004$

EX :- Let $S = \{1,2,3,4\}$, $P_i = \frac{1}{4}$, $A = \{1,2\}$, $B = \{2,3\}$, are (A) and (B) independent?

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = P(2) = \frac{1}{4}$

$P(A \cap B) \overset{?}{=} P(A)\, P(B)$

$\frac{1}{4} \overset{?}{=} \frac{1}{2} \times \frac{1}{2}$

$\frac{1}{4} \overset{\checkmark}{=} \frac{1}{4}$ $\therefore$ Yes they are independent.

ex:- let $S = \{1,2,3,4\}$, $A = \{1,2\}$

$B = \{2,3\}$, $P(\{1\}) = \frac{1}{2}$, $P(\{2\}) = \frac{1}{4}$

$P(\{3\}) = \frac{1}{8}$, $P(\{4\}) = \frac{1}{8}$, Are A and B S.I?

*statistically independent.

$P(A) = P(\{1,2\}) = P(\{1\} \cup \{2\})$ as $\{1\}$ & $\{2\}$ are disjoint.

$= P(\{1\}) + P(\{2\})$

$= \frac{1}{2} + \frac{1}{4}$

$= \frac{3}{4}$

$P(B) = P(\{2\} \cup \{3\}) = P\{(2)\} + P(\{3\})$

$= \frac{1}{4} + \frac{1}{8}$

$= \frac{3}{8}$

$P(A \cap B) = P(\{2\}) = \frac{1}{4}$

$P(A \cap B) \overset{?}{=} P(A)P(B)$

$\frac{1}{4} \overset{?}{=} \frac{3}{8} \times \frac{3}{4} \Rightarrow \frac{1}{4} \neq \frac{9}{32} \Rightarrow$ so A & B Are Not SI.

Ex:- $S = \{1,2,3,4\}$ $A = \{1,2\}$, $B = \{1,3\}$, $C = \{1,4\}$.

Are A, B and C, statistically independent?

$P(A) = \frac{1}{2}$

$P(B) = \frac{1}{2}$ $\Rightarrow$ $\boxed{P(A \cap B \cap C) \overset{?}{=} P(A)P(B)P(C)}$

$P(C) = \frac{1}{2}$

$P(A \cap B \cap C) = P(\{1\}) = \frac{1}{4}$ $\quad \frac{1}{4} \overset{?}{=} \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$P(A \cap B) \overset{?}{=} P(A)P(B)$

$P(A \cap C) \overset{?}{=} P(A)P(C)$ $\quad \frac{1}{4} \neq \frac{1}{8}$

$P(B \cap C) \overset{?}{=} P(B)P(C)$ So they are Not SI.

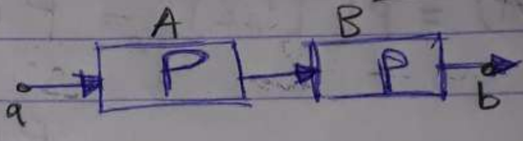
شرط واحد

يخرب كل الـ independence.

ex:- ~~Reli~~ Reliability of $\boxed{\text{a series}}$ system:- .

Suppose that a system is made up of two components connected in series, each component has a probability (P) of working "reliability". What is the probability that the system works assuming that components work independently

$\therefore P(A \cap B) = P(A)\,P(B)$
$= P * P$ ← [illegible]
$= P^2$

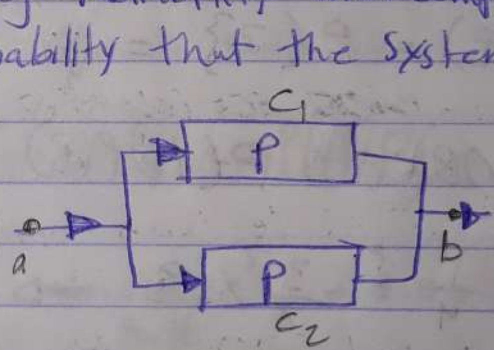


$P(A \cap B) = (0.9)^2 = 0.81$ ← $P = 90\%$ مثلاً إذا كانت

ex:- Reliability of a $\boxed{\text{Parallel}}$ system.

Suppose that a system is made up of two components connected in parallel. The system works if at least one component works properly. If each component has a probability (P) of working "Reliability" and components work independently. Find the probability that the system works.

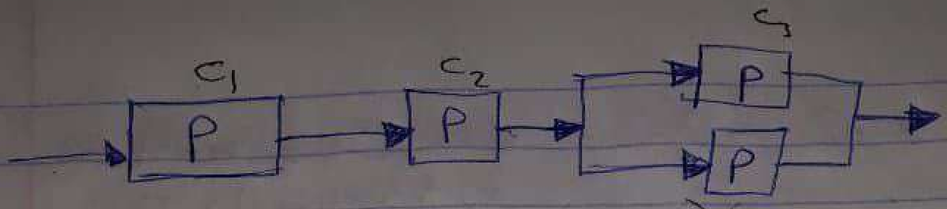
$\therefore P(R) = P(C_1 \overset{\text{or}}{\cup} C_2)$
$= P(C_1) + P(C_2) - \overset{\text{independent}}{P(C_1 \cap C_2)}$
$= 0.9 + 0.9 - (0.9)^2$
$= 0.99$



ومن هذا الجواب نلاحظ أنّ الـ P للـ parallel أعلى من series وهذا منطقي جداً ، وهذا يعني أنّ الـ cost لـ c_1 و c_2 بالتوازي ستكون أعلى من التوالي ، لأنه بها مضخات تكون قوية ، ولا يوجد بها مضخة قبلها أو بعدها تعتمد عليها كما في الحال تبع الـ series.

ex: "Mixed System", Find the reliability of the shown mixed system, assuming that all components work independently, and "P" is the ~~Probability~~ reliability (probability of working) of each component.

الرسمة ←



$$P(R) = P(C_1 \cap C_2 \cap [C_3 \cup C_4])$$

بما أنهم SI

$$= P(C_1)\,P(C_2)\,P(C_3 \cup C_4)$$

$$= P(C_1)\,P(C_2)\,[P(C_3) + P(C_4) - P(C_3 \cap C_4)]$$

*نستنتج أنّ وسط ما يكون series ← and تقاطع $\cap$ ، و parallel ← OR اتحاد $\cup$

Example: A coin may be fair or it may have two heads. We toss it (n) times and it comes up heads on each occasion. If our initial judgment was that both options for the coin (fair or both sides heads) were equally likely (Probable), what is our revised judgment in the light of the data?

طلبنا نفترض إنو $\boxed{2 = n}$ يعني رمينا الـ coin فقط مرتين.

Let A: event representing coin is fair. (H)(T)

B: event representing coin with Two heads (H)(H)

C: outcome of the experiment $\underbrace{HHHH\ldots H}_{n \text{ times}}$

فرضنا n=2

فهو بيسألني إذا رميت الكوين مرتين وطلع بالمرتين Heads

إيش احتمالية إنو يكون الـ coin هو fair؟ يعني $P(A|C)$

الوجه $P(H|A) * P(H|A)$

وما بنعرفها

$$\therefore P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)\,P(C|A)}{\text{الاحتمالية إنو يطلع fair أو يطلع two heads الكوين}} = \frac{P(A)\,P(C|A)}{P(C \cap A) + P(C \cap B)}$$

أو = union "U" = "+"

$$= \frac{P(A)\,P(C|A)}{P(A)\,P(C|A) + P(B)\,P(C|B)} = \frac{0.5 * (0.5 * 0.5)}{0.5 * (0.5)^2 + 0.5 * (1)^2}$$

إذا كان الكوين fair، احتمالية إنو يطلع معي 2 heads $(0.5 * 0.5)$

احتمالية إنو يكون 2 heads

إذا كان الـ coin ← two heads أو احتمالية {B} وطلع 2 heads $(1)^2$

$$=\frac{(0.5)^2}{(0.5)^2+0.5} = 0.2 = 20\% \Rightarrow$$ يعني احتمالية إني أرمي ال coin مرتين، إيش احتمالية إنو يكون ال coin fair فهي 20%.

إذا كانت $n=3$ فقط ما بيتغير في الحل هي القوى :-

$$= \frac{(0.5)^3}{(0.5)^3+0.5} = 11.1111\%$$

بنلاحظ إنو ال probability لل fair قلّت في كل مرة بنرمي ال coin فيها رح يطلع head، فهيك بتتعزز ثقتي إنو ال coin هو Two heads.

فلما نضل نزيد قيمة n لحد ما نوصل ∞ رح يصير الوضع :-

$$= \frac{(0.5)^n}{(0.5)^n+0.5} \Rightarrow$$ يعني أنه إذا أخذنا ال limit $n \to \infty$، رح يصير الجواب = صفر.

وهاد مراد السؤال، وهيك بنصير متأكدين إنو ال coin هو 100% two heads، و 0% fair.

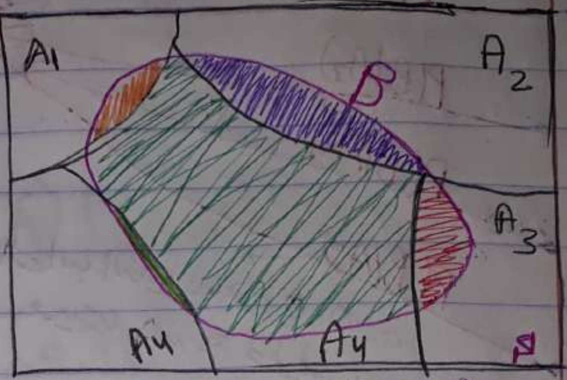
* ملاحظة هامة :- $P(A|C) = \frac{P(A\cap C)}{P(C)}$، ما بنقدر نحكي إنو $P(A)P(C) = P(A\cap C)$ لأنو هدول (A و C) مش S.I statically independent

يعني أنّ حدوث A مرتبط بحدوث C، وكل واحد يعتمد على الآخر، لهذا نحكي إنها $= P(A)\,P(C|A)$ فقط كل سبيل تغيير الشكل و المحاولة في حل السؤال حسب المعطى لأنه لم يذكر شيئاً بخصوص $P(A\cap C)$

* ملاحظة هامة :- في المقام :- $P(C\cap A) \cup P(C\cap B)$، حطينا ال Union زائد عشانه مستحيل يطلع $(C\cap A)$ و $(C\cap B)$ يكونوا joint، لأنو بالله كيف بدو يكون ال coin fair وبنفس الوقت is two heads فعشانهم Mutually exclusive (disjoint) حطيناه زائد (ال U) حسب القانون اللي أخذناه قبل.

Theorem of Total Probability

$S = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$



a sample space

$A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, ...

$\therefore A_1, A_2, A_3, A_4, A_5$ are disjoint

$P(B) = ??$, given $P(B/A_1)$, $P(B/A_2)$, $P(B/A_3)$,

$P(B/A_4)$, $P(B/A_5)$, $P(A_1)$, $P(A_2)$, $P(A_3)$, $P(A_4)$, $P(A_5)$.

$$P(B) = P([B \cap A_1] \cup [B \cap A_2] \cup [B \cap A_3] \cup [B \cap A_4] \cup [B \cap A_5])$$

They are disjoint

[illegible]: $[B \cap A_1] \cap [B \cap A_2] = B \cap [\overset{\phi}{A_1 \cap A_2}]$

$= B \cap \phi$

$= \phi$

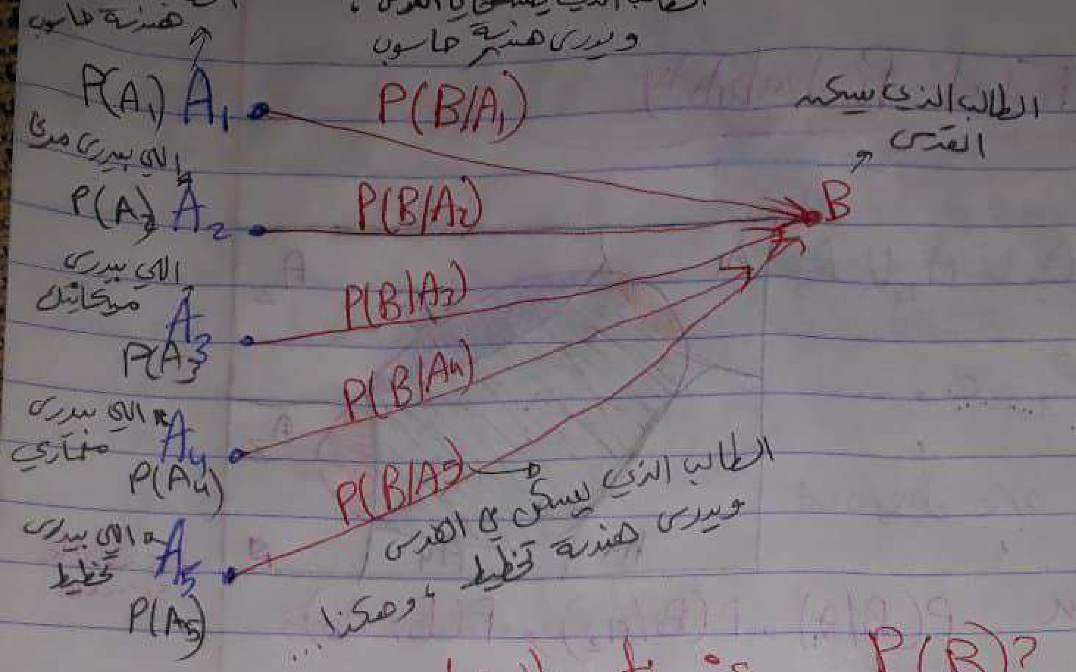
$$= P(B \cap A_1) + P(B \cap A_2) + P(B \cap A_3) + P(B \cap A_4) + P(B \cap A_5)$$

$$= P(A_1)P(B/A_1) + P(A_2)P(B/A_2) + P(A_3)P(B/A_3) + P(A_4)P(B/A_4) + P(A_5)P(B/A_5).$$

$$= \sum_{i=1}^{5} P(A_i)P(B/A_i)$$

تكملة

مثلاً :

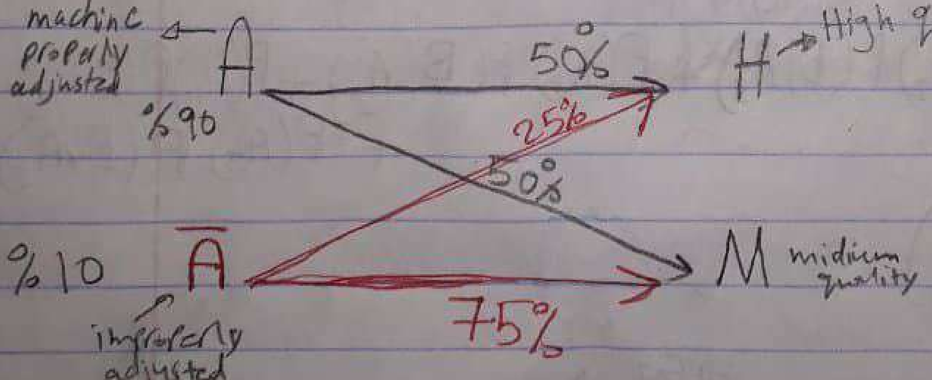


What is $P(B)$?

$$P(B) = P(A_1)P(B|A_1) + \dots + P(A_5)P(B|A_5)$$

والمفروض يعطيني قيم لـ $P(A_i)$ و $P(B|A_i)$ ، ومطلوب $P(B)$

Example:- Suppose that when a machine is adjusted properly (A), 50% of the items produced by it are of high quality and the other 50% are of medium quality. Suppose, however, that the machine is improperly adjusted ($\bar{A}$) during 10% of the time and that under these conditions 25% of the items produced by it are high quality and 75% are of medium quality.



(a) Suppose that one item produced by the machine is selected at random, find the probability that it is of medium quality.

$P(M) = ??$

$$P(M) = P(A)\,P(M/A) + P(\bar{A})\,P(M/\bar{A})$$

$$= (0.9)(0.5) + (0.1)(0.75)$$

$= 0.525$

$= 52.5\%$

(b) If one item is selected at random, and found to be of medium quality, what is the probability that the machine was adjusted properly.

مش معطى — معطى

$$P(A/m) = \frac{P(A \cap m)}{P(m)} = \frac{P(m/A)\,P(A)}{P(m)} = \frac{(0.5)(0.9)}{0.525}$$

$= 0.85714$

$= 85.714\%$

بعرف بسهولة إنه:

مطلعناها بفرع (a)

$$P(A \cap m) = P(A)\,P(m/A) = P(m)\,P(A/m)$$

multiplication Rule — وهي ال

فناخد هاي دايما ~ كل معطياتها معنا

ومستبدل هالحكي أستخدم نظرية جديدة :D

Baye's Theorem

If $A_1, A_2, A_3, \dots A_n$ are disjoint events defined on (S), and (B) is another event defined on (S) (Same condition as Total Probability theorem), then :-

معطى بيكون

$$P(A_j/B) = \frac{P(A_j)\,P(B/A_j)}{\sum^{n} P(A_i)\,P(B/A_i)} = \frac{P(A_j \cap B)}{P(B)}$$

هنا بنطلع بال Theorem of Total Probability

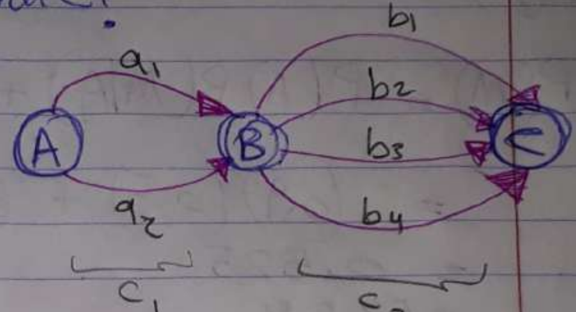
Multiplication Rule :-

Ex) There are two roads between A and B, and four roads between B and C. How many different roads can one travel between A and C?

عدد ال roads   عدد A

$\therefore\ n = 2 \times 4 = 8$

عدد b



Graph → $a_1\ b_1$, $a_1\ b_2$, $a_1\ b_3$, $a_1\ b_4$, $a_2\ b_1$, $a_2\ b_2$, $a_2\ b_3$, $a_2\ b_4$ عدد الطرق 8

* مابزبط أستخدم ال Multiplication Rule

عندما تكون ال 2 processes dependent

يعني إنو يحطلي شروط على الطرق، مثلاً إذا مشيت من a_1 فقط بزبط معاك تمشي من b_1 و b_2 ، أو إذا مشيت من a_2 فقط بزبط معاك تمشي من b_2, b_3 .

Permutation

1) Sampling without replacement (repetition is not allowed)

① ② ③ ④

we want to pick 2 balls, one at a time, without replacement, what is the total number of options available? $\underline{3} \times \underline{4} = 12$

12 خيار → 12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43.

2) Sampling with replacement. (repetion is allowed)

$\underline{4} \times \underline{4} = 16$ → 11, 12, 13, 14, 21, 22, 23, 24, 31, 32, 33, 34, 41, 42, 43, 44.

سؤال نفس 1 بس عادي أرجع الكرة للسلة

16 خيار

Ex 3- How many different five-letter computer passwords can be formed:-

[a] If a letter can be used more than once.
We have 26 letters in the alphabet:-

$$N = \frac{26}{5} \times \frac{26}{4} \times \frac{26}{3} \times \frac{26}{2} \times \frac{26}{1} = (26)^5$$

5 أماكن → حسب السؤال

[b] If each word contains each letter no more than once.
يعني أحرفنا فش تكرار .

$$\therefore n = \frac{26}{1} \times \frac{25}{2} \times \frac{24}{3} \times \frac{23}{4} \times \frac{22}{5}$$

نفسها

$\Rightarrow n = 26 \rightarrow$ العدد الكلي ، $k = 5 \rightarrow$ عدد المرات

$$\therefore P^n_k = \frac{n!}{(n-k)!} = \frac{26!}{(26-5)!} = \frac{26 \times 25 \times 24 \times 23 \times 22 \times \cancel{21}}{\cancel{(21)!}}$$

$$= 26 \times 25 \times 24 \times 23 \times 22$$

هاي الي اسمها "تباديل" بالعربي ، أخذناها بصف 11 .

[c] What is the probability of making a password such that each letter is used only once?

$$P = \frac{\text{each letter is selected only once}}{\text{all options (with replacement)}}$$

$$= \frac{26 \times 25 \times 24 \times 23 \times 22}{(26)^5}$$

Ex) An apartment building has eight floors (numbered 1 to 8). If seven people get on the elevator on the first floor, what's the probability that:-

[a] All get of on different floors?

Number of points in the sample space:-
First person can get off at any of the 7 floors.

Person (2) can get of at any of the 7 floors.
يعني كل شخص independent عن الآخر

→ The number of ways people can get off :-

$$N = \underbrace{7}_{1} \times \underbrace{7}_{2} \times \underbrace{7}_{3} \times \underbrace{7}_{4} \times \underbrace{7}_{5} \times \underbrace{7}_{6} \times \underbrace{7}_{7}$$
$$= 7^7$$

→ Here the problem is to find the number of permutations of 7 objects taking 7 at a time.

$$\therefore P = \frac{\text{each floor selected only once}}{\text{all options (with replacement)}}$$

$$= \frac{7!}{7^7} = 6.119 \times 10^{-3}$$

[b] Here there 7 ways whereby all seven persons get off on the same floor.

$$P = \frac{7 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1}{7^7} \leftarrow$$

كان الشخص الأول عنده خيار ينزل على أي واحد من السبع طوابق ، بس باقي الأشخاص رح ينزلوا معه إذن ما عندهم خيار ف (1) .

$$= \frac{7}{7^4}$$

Combinations :- توافيق

(order is NOT important)

If we have a basket of balls, and we want to pick $\boxed{2}$:-

① ② ③ ④

$*$ with replacement $\left(\begin{matrix}\text{order is}\\ \text{important}\end{matrix}\right) = 4 \times 4 = 4^2$

$*$ without replacement $\left(\begin{matrix}\text{order is}\\ \text{important}\end{matrix}\right) = 4 \times 3 = 12 = P_2^4$

$n=4$, $k=2$

$*$ without replacement $\left(\begin{matrix}\text{order is}\\ \text{Not important}\end{matrix}\right) = C_{k=2}^{n=4} = \binom{n}{k}$

$$= \binom{4}{2} = \frac{P_k^n}{k!} = \frac{4!}{(4-2)! * 2!} = \frac{\overset{2}{\cancel{4}} \times 3 \times \cancel{2!}}{\cancel{2!} \times \cancel{2} \times 1} = 6$$

معناها إنو فش فرق بين $\boxed{12 \text{ و } 21}$.

$b_1 \backslash b_2$	1	2	3	4
1	~~1,1~~	(1,2)	(1,3)	(1,4)
2	~~2,1~~	~~2,2~~	(2,3)	(2,4)
3	~~3,1~~	~~3,2~~	~~3,3~~	(3,4)
4	~~4,1~~	~~4,2~~	~~4,3~~	~~4,4~~

6

$*$ without replacement (order is important)

$*$ without replacement (order is not important)

عندنا إذا (1,4) أو (4,1) مش فارقة ، المهم وهو مهم لأنو القصد مش مهم الترتيب المهم بالآخر طلع بإيدي الكرة 4 و الكرة 1 ، ومش مهم مين طلع أول .

Ex From four persons (set of elements), how many committees (subsets) of two members (elements) may be chosen?

Let the persons be identified by the initials A, B, C, and D.

subsets are :- (A,B), (A,C), (A,D), (B,C), (B,D), (C,D).

$$N = \binom{4}{2} = \frac{4!}{2!.(4-2)!} = 6 .$$

$\Longrightarrow$

Missing sequences : (A, A), (B, B), (C, C), (D, D)

repetition is Not allowed.

Missing sequences : (B, A), (C, A), (D, A), (C, B), (D, B), (D, C)

order is not important.